Constructing Faithful Maps over Arbitrary Fields

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joint work with

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Algebraic Independence

Definition: Algebraic Independence

A given set of polynomials $\{f_1, f_2, \ldots, f_m\} \subseteq \mathbb{F}[x_1, x_2, \ldots, x_n]$ is said to be algebraically dependent if there is a non-zero polynomial combination of these that is zero.

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Question: Can we test algebraic independence efficiently?

Checking Algebraic Independence

Working with Annihilating Polynomials [Kay09, GSS18]

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For $f_1, f_2, \ldots, f_m \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ and $\mathbf{f} = (f_1, f_2, \ldots, f_m)$,

$$\mathbf{J_x}(\mathbf{f}) = \begin{bmatrix} \partial_{x_1}(f_1) & \partial_{x_2}(f_1) & \ldots & \partial_{x_n}(f_1) \\ \partial_{x_1}(f_2) & \partial_{x_2}(f_2) & \ldots & \partial_{x_n}(f_2) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1}(f_m) & \partial_{x_2}(f_m) & \ldots & \partial_{x_n}(f_m) \end{bmatrix}$$

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The Jacobian Criterion [Jac41]

If $\mathbb{F}$ has characteristic zero, $\{f_1, f_2, \ldots, f_m\}$ is algebraically independent if and only if its Jacobian matrix is full rank.

Rank Preserving Maps

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Given a set of polynomials $\{f_1, f_2, \ldots, f_m\}$ with algebraic rank k, a map

$$\varphi : \{x_1, x_2, \ldots, x_n\} \to \mathbb{F}[y_1, y_2, \ldots, y_k]$$

is said to be a faithful map if the algebraic rank of $\{f_1(\varphi), f_2(\varphi), \ldots, f_m(\varphi)\}$ is also k.

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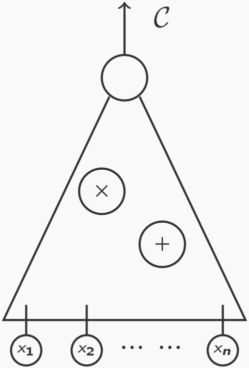
Bonus: Helps in polynomial identity testing.

Faithful Maps and Poly. Identity Testing [BMS11, ASSS12]



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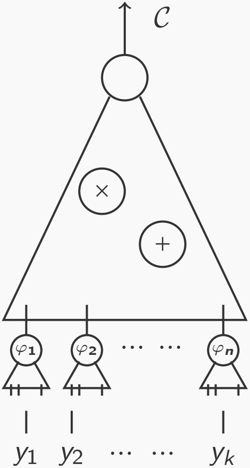

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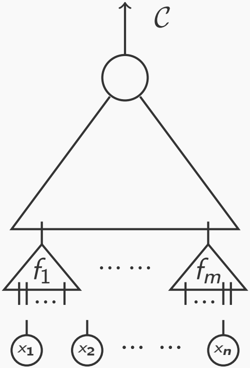

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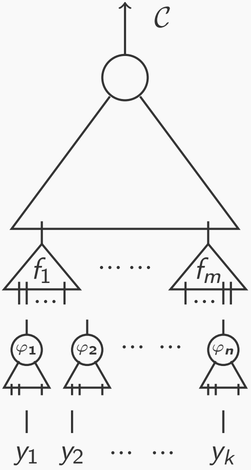

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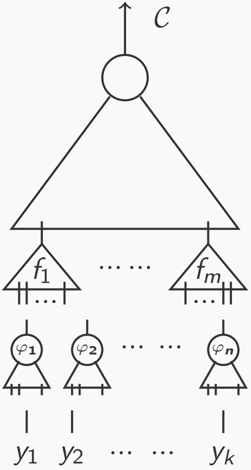

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$$\mathcal{C}(f_1, f_2, \ldots, f_m) \neq 0 \text{ if and only if}$$

$$(\mathcal{C}(f_1(\varphi), f_2(\varphi), \ldots f_m(\varphi))) \neq 0.$$

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This work: Construct Faithful Maps over arbitrary fields and extend results in [ASSS12] to other fields.

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2. M_φ preserves rank

A Rank Preserving Matrix and a Faithful Map [BMS11]

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Finite Characteristic: Entries in "inverse" have denominators that are partial derivatives of some annihilators, which can become zero.

Looking Further in the Taylor Expansion [PSS16]

For any $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ and $\mathbf{z} \in \mathbb{F}^n$,

$$f(\mathbf{x} + \mathbf{z}) - f(\mathbf{z}) = \underbrace{x_1 \cdot \partial_{x_1} f + \cdots + x_n \cdot \partial_{x_n} f}_{\text{Jacobian}} + \text{higher order terms}$$

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Definition: A new Operator

For any $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$,

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$$\hat{\mathcal{H}}(\mathbf{f}) = \begin{bmatrix} \ldots & \mathcal{H}_t(f_1) & \ldots \\ \ldots & \mathcal{H}_t(f_2) & \ldots \\ & \vdots & \\ \ldots & \mathcal{H}_t(f_k) & \ldots \end{bmatrix}.$$

The [PSS16] Criterion

A given set of polynomials $\{f_1, f_2, \ldots, f_k\} \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ is algebraically independent if and only if for a random $\mathbf{z} \in \mathbb{F}^n$, $\{\mathcal{H}_t(f_1), \mathcal{H}_t(f_2), \ldots, \mathcal{H}_t(f_k)\}$ are linearly independent in

$$\frac{\mathbb{F}(\mathbf{z})[x_1, x_2, \ldots, x_n]}{\mathcal{I}_t}$$

where t is the inseparable degree of $\{f_1, f_2, \ldots, f_k\}$ and

$$\mathcal{I}_t = \langle \mathcal{H}_t(f_1), \mathcal{H}_t(f_2), \ldots, \mathcal{H}_t(f_k) \rangle_{\mathbb{F}(\mathbf{z})}^{\geq 2} \bmod \langle \mathbf{x} \rangle^{t+1} \subseteq \mathbb{F}(\mathbf{z})[\mathbf{x}].$$

Alternate Statement for the [PSS16] Criterion

$\{f_1, f_2, \ldots, f_k\}$ is algebraically independent if and only if for every $(v_1, v_2, \ldots, v_k)$ with v_is in $\mathcal{I}_t$,

$$\mathcal{H}(\mathbf{f}, \mathbf{v}) = \begin{bmatrix} \ldots & \mathcal{H}_t(f_1) + v_1 & \ldots \\ \ldots & \mathcal{H}_t(f_2) + v_2 & \ldots \\ & \vdots & \\ \ldots & \mathcal{H}_t(f_k) + v_k & \ldots \end{bmatrix} \text{ has full rank over } \mathbb{F}(\mathbf{z}).$$

The Goal

What we know:

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What we want to show:

$$\mathcal{H}(\mathbf{f}(\varphi),\mathbf{u}) = \begin{bmatrix} \dots & \mathcal{H}_t(f_1(\varphi)) + u_1 & \dots \\ \dots & \mathcal{H}_t(f_2(\varphi)) + u_2 & \dots \\ & \vdots & \\ \dots & \mathcal{H}_t(f_k(\varphi)) + u_k & \dots \end{bmatrix}$$

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Sufficient Properties

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2. $\mathcal{H}(\mathbf{f}(\varphi), \mathbf{v}(\varphi)) = \varphi(\mathcal{H}(\mathbf{f}, \mathbf{v})) \times M_\varphi$: Chain Rule
3. $\text{rank}(\mathcal{H}(\mathbf{f}, \mathbf{v})) = \text{rank}(\varphi(\mathcal{H}(\mathbf{f}, \mathbf{v})))$: a_is are responsible for this
4. M_φ preserves rank

The Matrix Decomposition

$$\begin{bmatrix} \mathcal{H}(\mathbf{f}(\varphi), \mathbf{v}(\varphi)) \end{bmatrix} = \overbrace{\begin{bmatrix} \varphi(\mathcal{H}(\mathbf{f}, \mathbf{v})) \end{bmatrix}}^{\text{labelled by } \mathbf{x}^{\mathbf{e}}} \times \underbrace{\begin{bmatrix} M_{\varphi} \end{bmatrix}}_{\text{labelled by } \mathbf{y}^{\mathbf{d}}}$$

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where

$$M_\varphi(\mathbf{x}^{\mathbf{e}}, \mathbf{y}^{\mathbf{d}}) = \begin{cases} \text{coeff}_{\mathbf{y}^{\mathbf{d}}}(\varphi(\mathbf{x}^{\mathbf{e}})) & \text{if } \sum e_i = \sum d_i \\ 0 & \text{otherwise} \end{cases}$$

What makes Vandermonde type matrices work?

$$\begin{bmatrix} \quad A \quad \end{bmatrix} \times \begin{bmatrix} \\ M \\ \\ \end{bmatrix} = \begin{bmatrix} AM \end{bmatrix}$$

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$\mathrm{wt}(x_i)$ is distinct for each i

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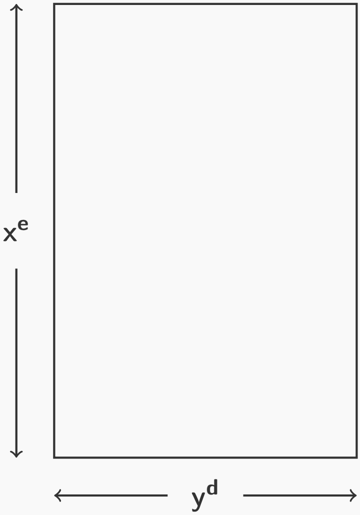

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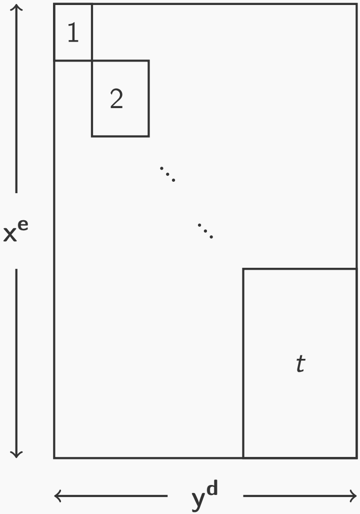

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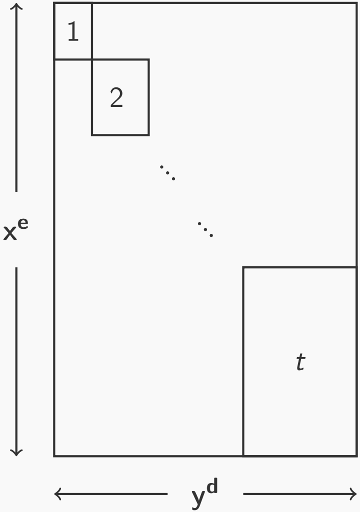

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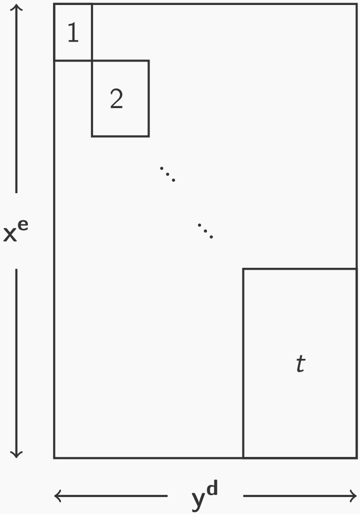

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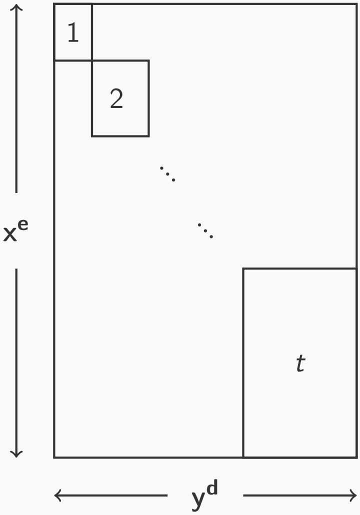

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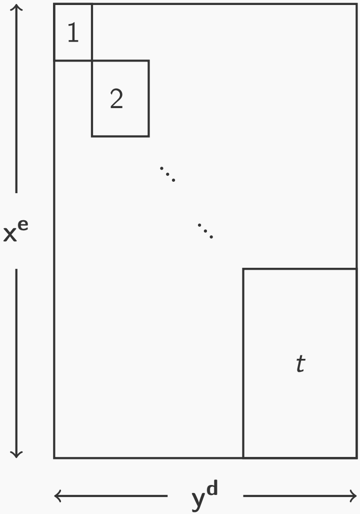

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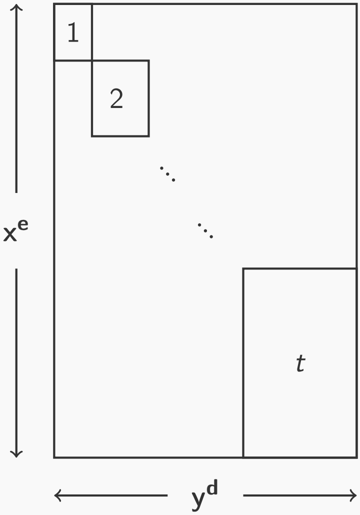

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A Rank Preserving Matrix

$$\begin{bmatrix} & & A & & \end{bmatrix} \times \begin{bmatrix} \\ \\ & M & \\ \\ \\ \end{bmatrix} = \begin{bmatrix} & AM & \end{bmatrix}$$

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$$k \left[\quad A \quad \right] \times \left[\quad M \quad \right] = \left[\quad AM \quad \right]$$

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What we want: k columns of AM that are linearly independent.

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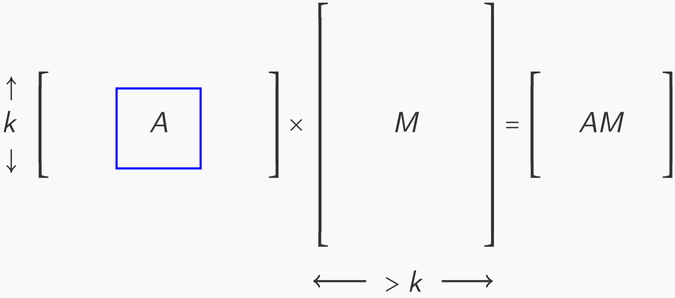

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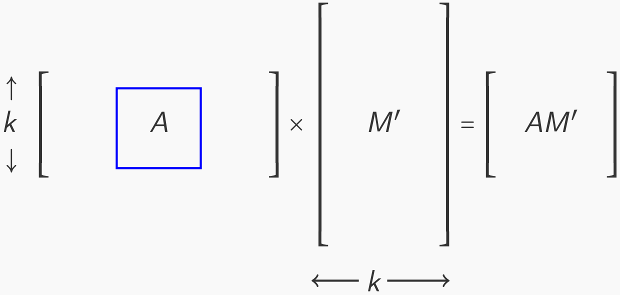

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- $M' \equiv k$ columns of M such that $\deg_s(\det(M'_{B_0})) = \mathrm{wt}(B_0)$

A few details

About $\deg_s(\det(M'_{B_0}))$ **for** $B \neq B_0$:

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About M'

- M' can always be chosen such that its columns are indexed by "pure" monomials.

A Faithful Map over Arbitrary Fields

$$\varphi : x_i \to \sum_{j=1}^{k} s^{\mathrm{wt}(i)j} y_j + a_i y_0 \text{ and } z_i \to \sum_{j=1}^{k} s^{\mathrm{wt}(i)j} w_j + a_i y_0$$

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An Instantiation

Theorem

Let $f_1, f_2, \ldots, f_m \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ *be s-sparse polynomials such that* $\text{algrank}(f_1, f_2, \ldots, f_m) = k$ *and the inseparable degree is* t*.*
If t *and* k *are bounded by a constant, then, there is an explicit deterministic construction of a faithful homomorphisms in* $\text{poly}(n, m, s)$ *time.*

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Explicit faithful homomorphisms can also be constructed efficiently for other models studied in [ASSS12] when we have similar inseparable degree bounds.

Open Threads

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Thank you!

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