EXPONENTIAL DEPTH-HIERARCHY FOR CONSTANT-DEPTH MULTILINEAR CIRCUITS

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Multilinear Polynomials

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Multilinear monomial $\prod_{i \in S} x_i \quad (S \subseteq [n])$

Equivalent x^S

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E.g: Determinant, Permanent

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− Not succinct.

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$\Sigma\Pi\Sigma\Pi$, $\Sigma\Pi\Sigma\Pi\Sigma$, ...

Constant-depth Algebraic formulas

$\Sigma\Pi\Sigma\Pi$

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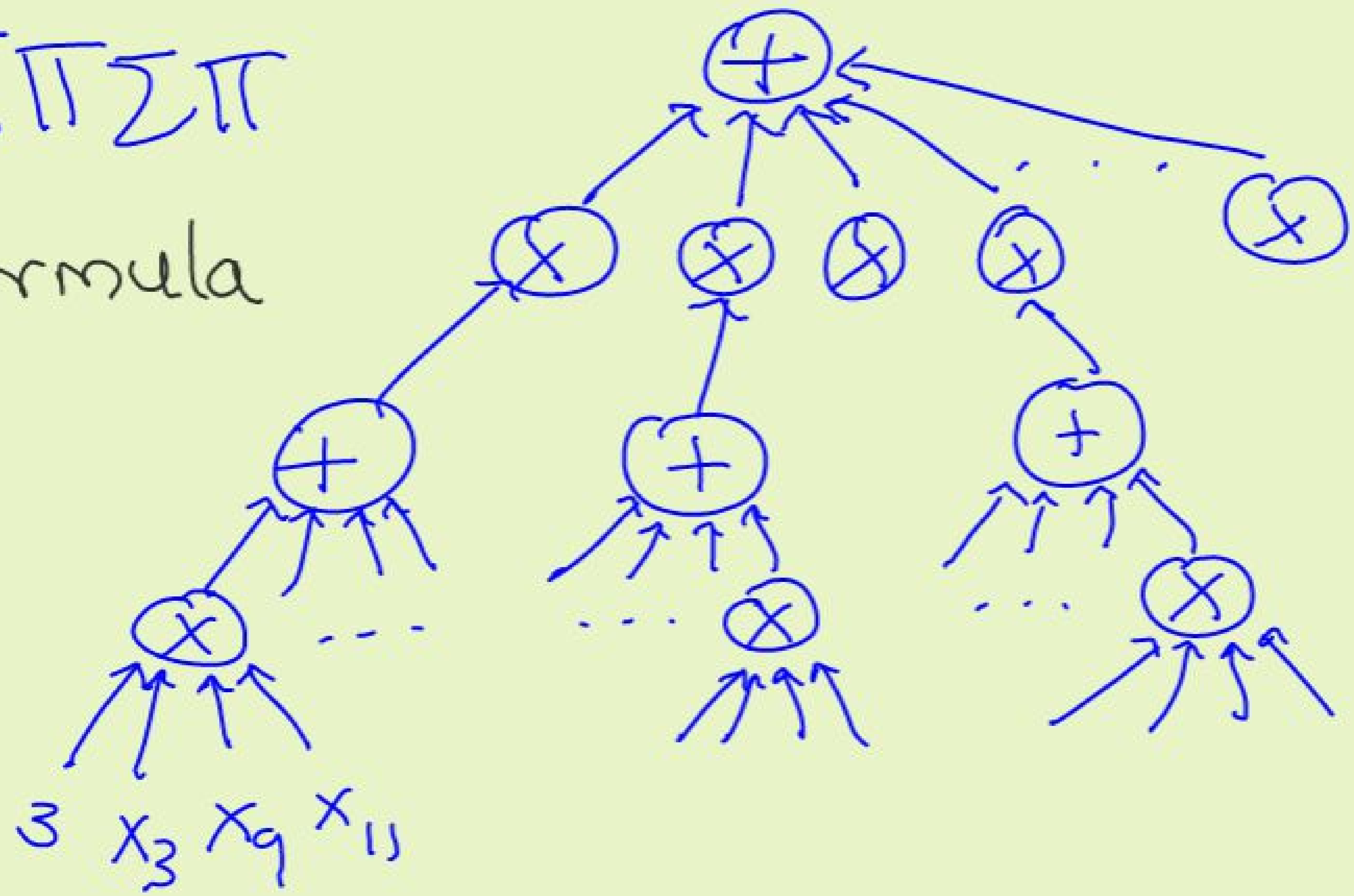

Constant-depth Algebraic formulas

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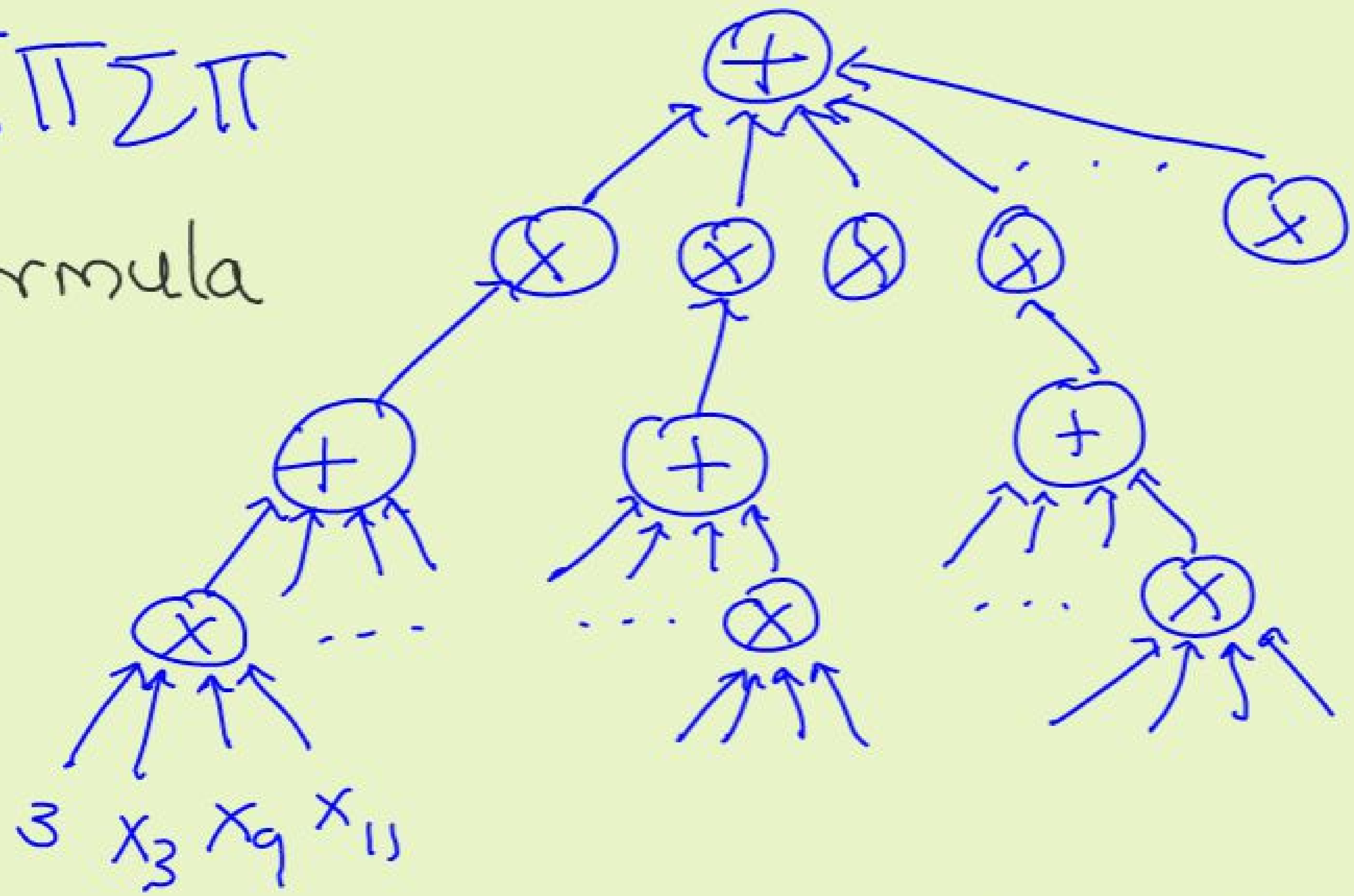

Size of expression = # of vertices in this tree

Constant-depth Algebraic formulas

$\Sigma\Pi, \Sigma\Pi\Sigma, \Sigma\Pi\Sigma\Pi, \Sigma\Pi\Sigma\Pi\Sigma, \ldots$

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Nesting measured by Product-Depth

Product-Depth = # of Π's in name of formula class.

Constant-depth Algebraic formulas

$$\underbrace{\Sigma\Pi,\ \Sigma\Pi\Sigma}_{\text{Product depth 1}},\ \underbrace{\Sigma\Pi\Sigma\Pi,\ \Sigma\Pi\Sigma\Pi\Sigma}_{\text{Product depth 2}},\ \ldots$$

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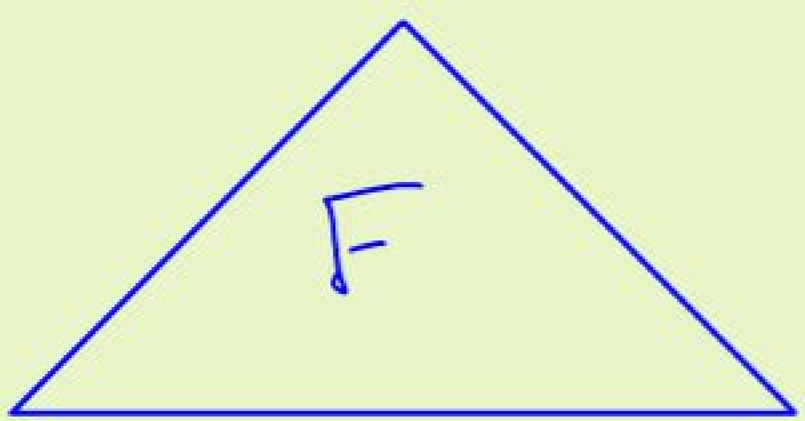

P.depth $\Delta+1$ formula of size s

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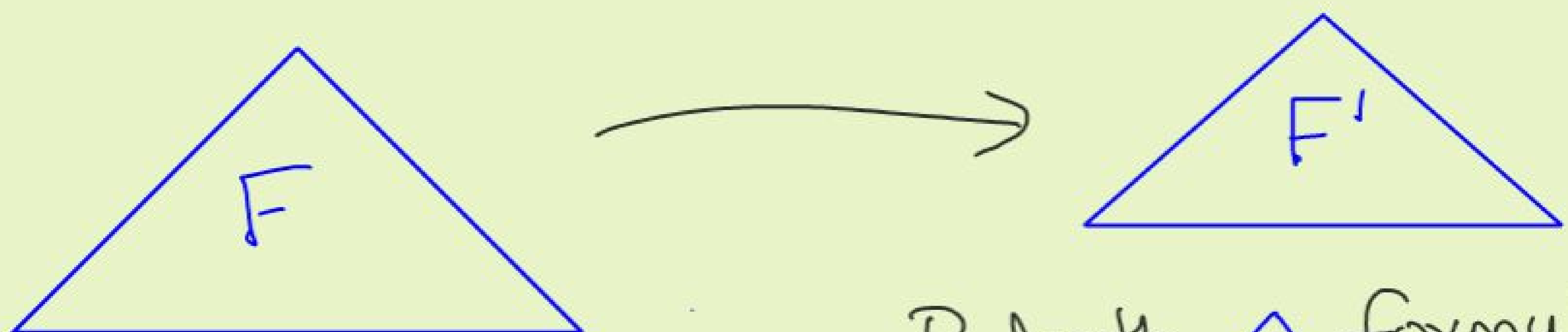

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P.depth Δ formula of size $s' = f(s)$

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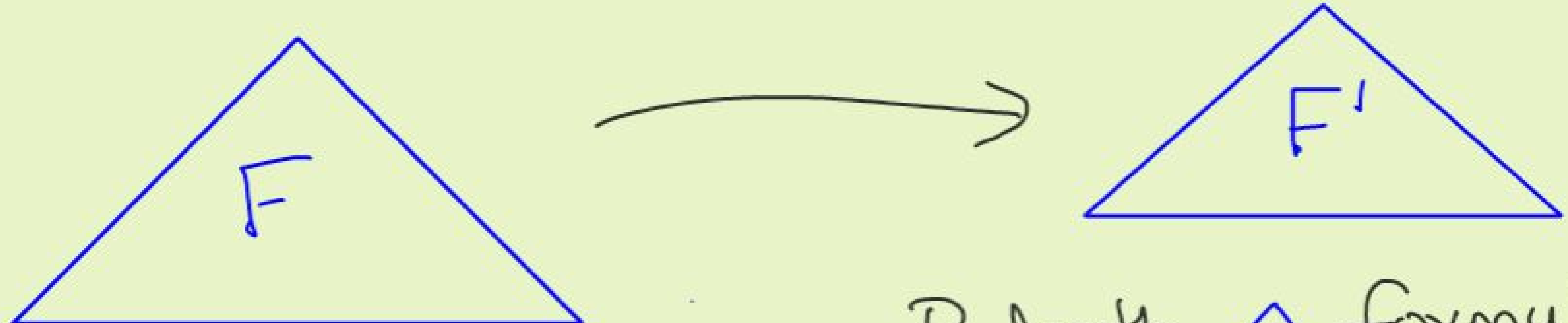

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Worst-case blowup?

Reducing p.depth by 1

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Distribute $\times$ over $+$.

$$\prod_{i=1}^{\ell} \Big(\sum_j T_{i,j} \Big) \xrightarrow[\substack{\text{blowup} \\ \exp(\ell)}]{} \sum_{j_1,\dots,j_\ell} T_{1,j_1} T_{2,j_2} \cdots T_{\ell,j_\ell}$$

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Depth $\Delta+1$: $\Sigma\Pi\Sigma \cdots \Sigma\underbrace{\Pi\Sigma}\Pi \cdots \Sigma\Pi\Sigma$

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Depth Δ: $\Sigma\Pi\Sigma \cdots \Sigma\Pi \cdots \Sigma\Pi\Sigma$

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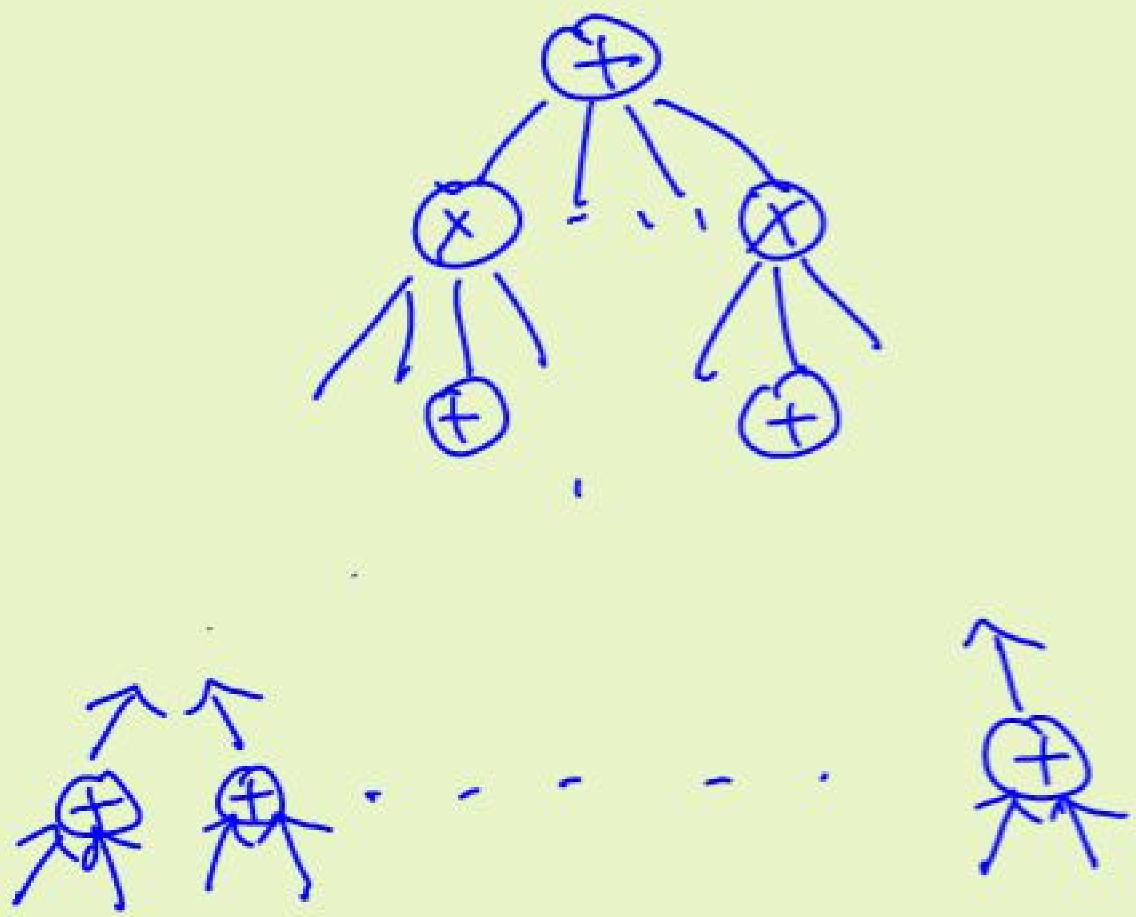

s vertices

& p.depth $\Delta+1$

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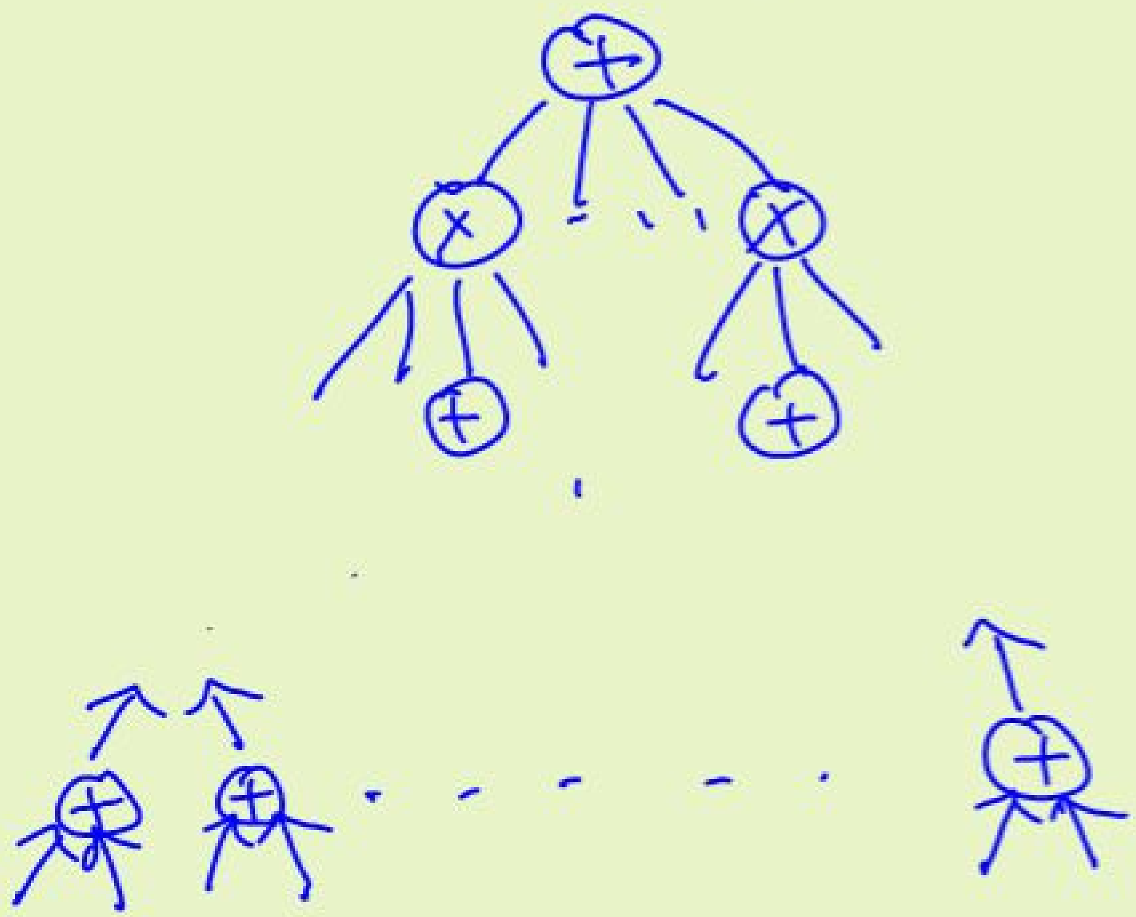

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$\Downarrow$

$\times$s of fan-in

$\leq s^{1/\Delta}$.

Reducing p.depth by 1

Fact: P.depth $\Delta+1$, size s

$\downarrow$

P.depth Δ, size $\exp(s^{1/\Delta + o(1)})$

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Δ	Blowup
1	$\exp(s)$
2	$\exp(\sqrt{s})$
3	$\exp(s^{1/3})$
$\vdots$	$\vdots$

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Is this exponential blowup necessary?

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All subexpressions compute multilinear polynomials.

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Eg:

$$\underbrace{(x_1+x_2+x_3)^2 - x_1^2 - x_2^2 - x_3^2}_{\text{Non-multilinear } \Sigma\Pi \text{ formula}} = \underbrace{2x_1x_2 + 2x_2x_3 + 2x_3x_1}_{\text{Multilinear } \Sigma\Pi \text{ formula}}$$

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Qn: Multilinear separation b/w depths

Δ & $\Delta+1$?

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→ Raz (2005), Raz-Shpilka-Yehudayoff (2007), Hrubeš-Yehudayoff (2009), Dvir et al. (2012), Kayal-Nair-Saha (2015) Kumar-Volk (2017) ...

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Δ	s_Δ
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→ Quasipolynomial depth hierarchy.

Raz-Yehudayoff

→ Quasipolynomial separation b/w p.depth Δ and p.depth $\Delta+1$

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→ Our result: Similar improvement for all constant Δ.

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Thm: For any const. Δ, there is an expli-
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$$\alpha_\Delta = \frac{1}{\Delta^{O(\Delta)}}.$$

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→ Håstad (1986): Exponential depth-hierarchy.

Constructing the hard polynomial

$P_{\Delta+1}$ → small formula of p.depth $\Delta+1$

$P_{\Delta+1}$ → large formulas of p.depth $\leq \Delta$

Graph Polynomials (ABPs)

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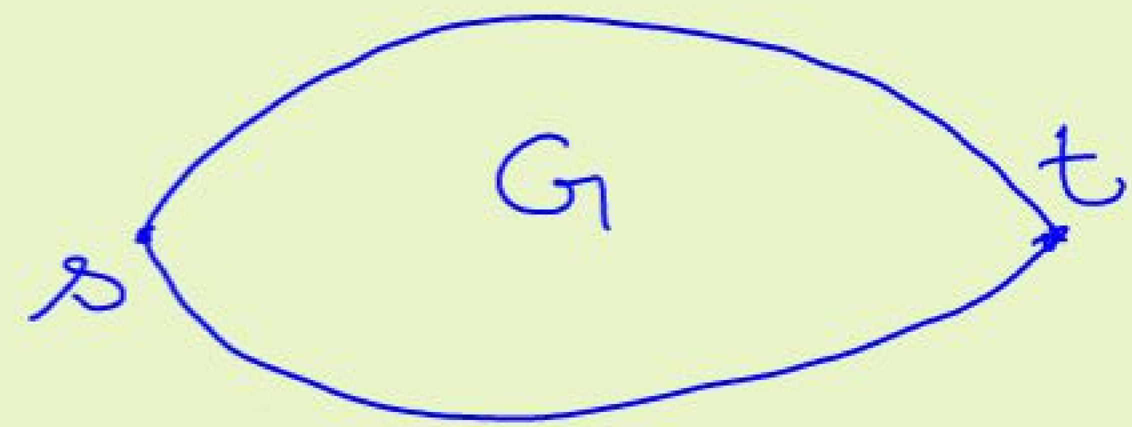
G - Directed Acyclic Graph (DAG)

Source s, Sink t



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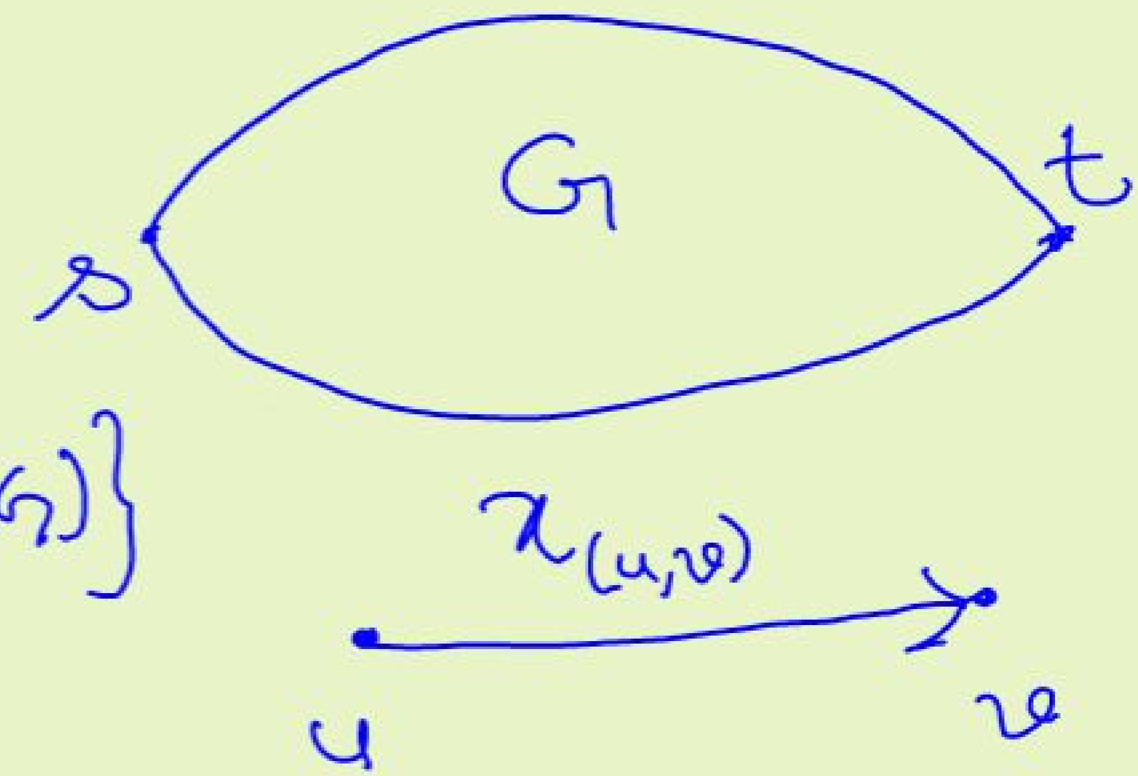
Variables $X = \{x_e \mid e \in E(G)\}$



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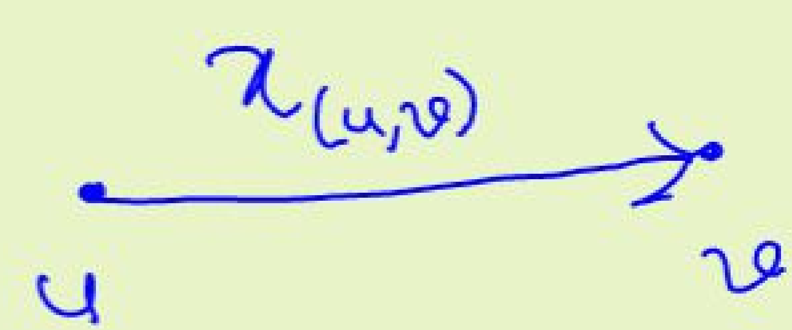

$$P_G = \sum_{\substack{\pi : s \rightsquigarrow t \\ \pi = (u_0 = s, \ldots, u_\ell = t)}} x_{u_0,u_1} x_{u_1,u_2} \cdots x_{u_{\ell-1},u_\ell}$$

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$$s \xrightarrow{x_1} u_1 \xrightarrow{x_2} u_2 \xrightarrow{x_3} u_3 \xrightarrow{x_4} t$$

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$$\begin{tikzcd}
 & u_1 \arrow[dr, "x_2"] & \\
s \arrow[ur, "x_1"] \arrow[dr, "x_3"'] & & t \\
 & u_2 \arrow[ur, "x_4"'] &
\end{tikzcd}$$

$$P_G = x_1 x_2 + x_3 x_4$$

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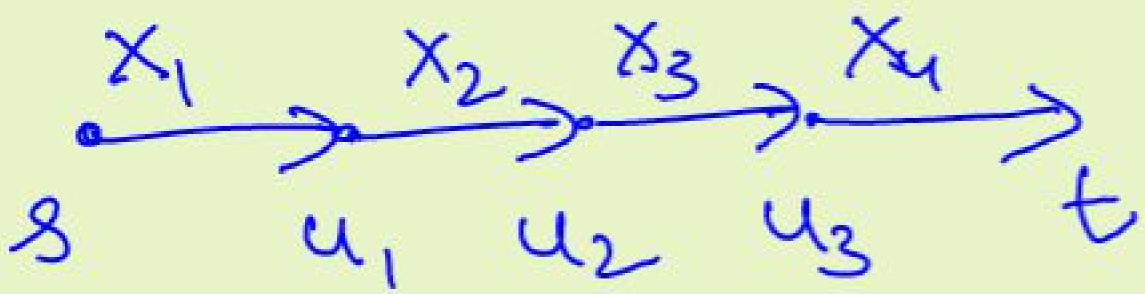

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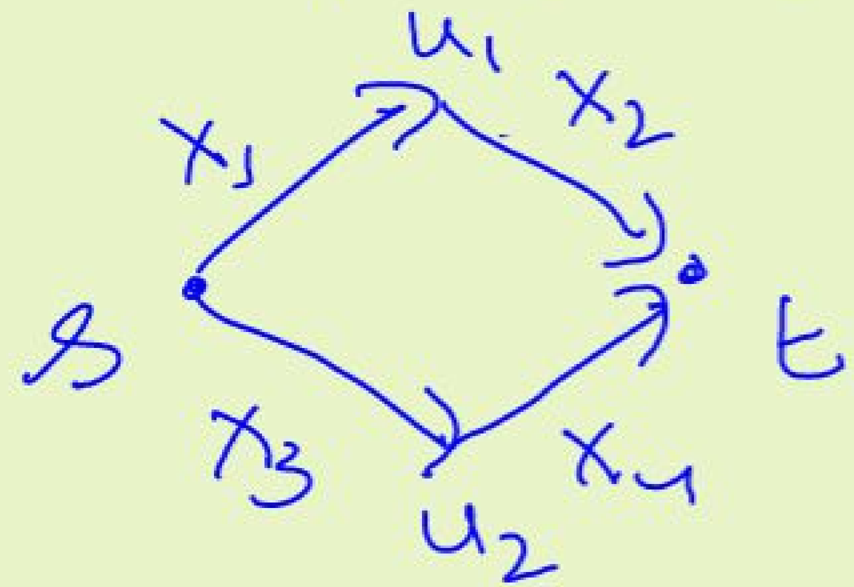

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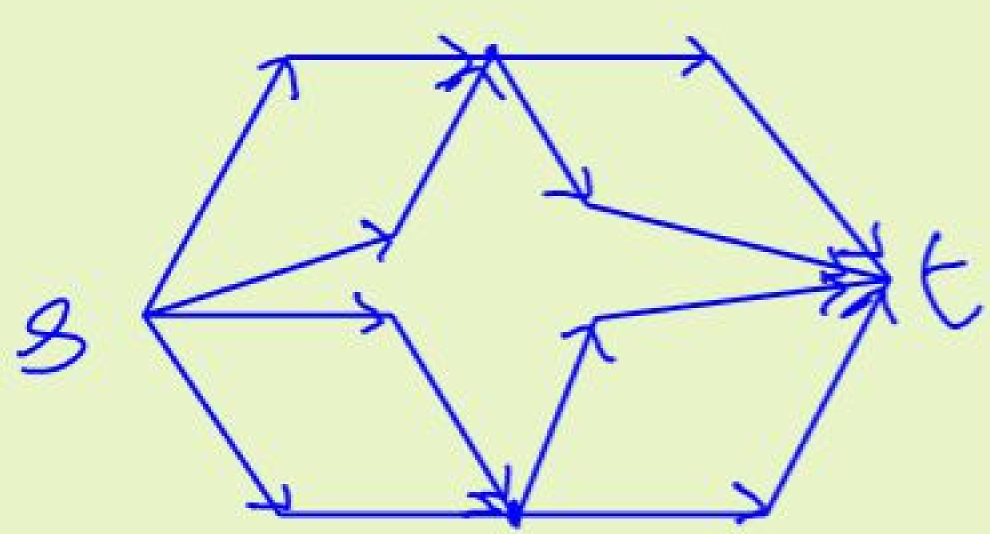

$P_G =$ sum over 8 paths of length 4

Composing graphs

$G_1, \ldots, G_r$ - Labelled DAGs

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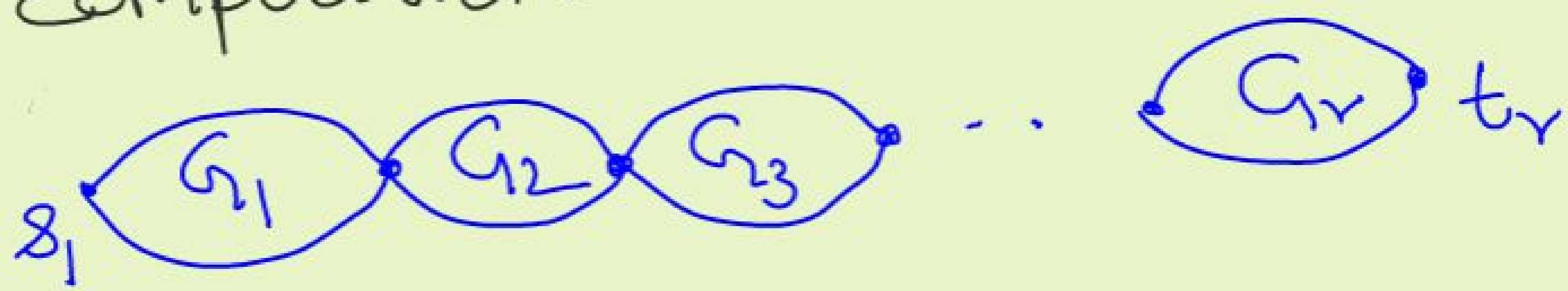
Series composition

$G_{ser} =$



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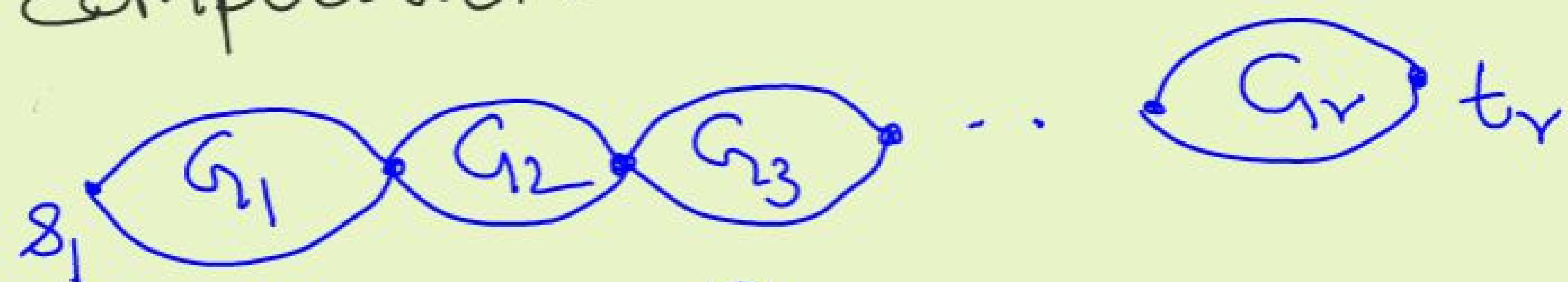
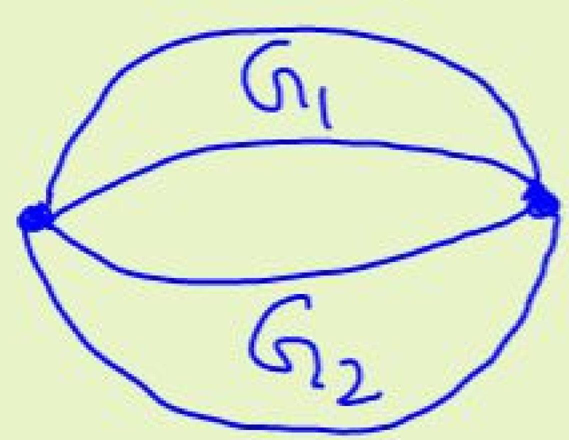
$G_{ser} =$ 

$P_{ser} = P_{G_1} \cdot P_{G_2} \cdot \ldots \cdot P_{G_r}$

Parallel composition

$G_{par} =$ 

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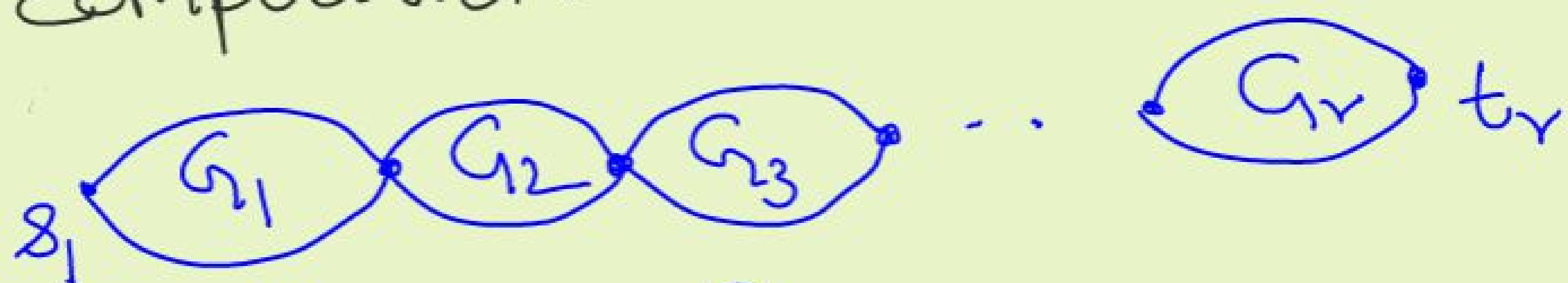
$G_{ser} =$ 

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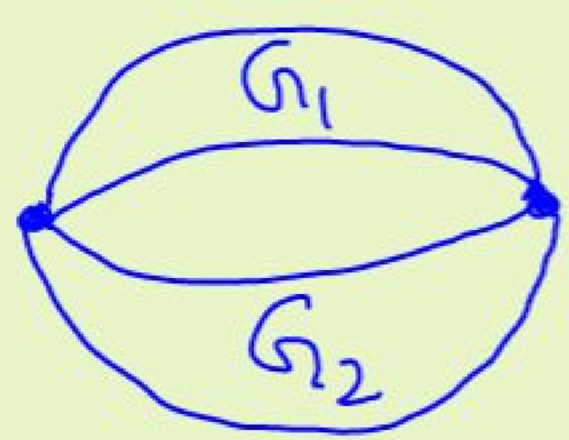
$G_{par} =$ 

$P_{par} = P_{G_1} + P_{G_2}$

The hard polynomials (Chen et al. 2016)

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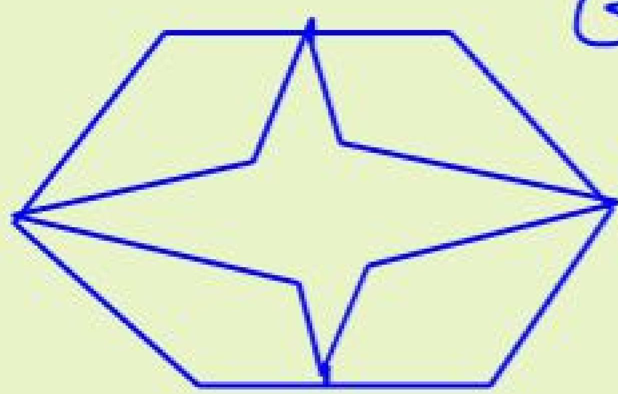

G_1

P_1 computed by $\Sigma\Pi$ formula of size $O(1)$.

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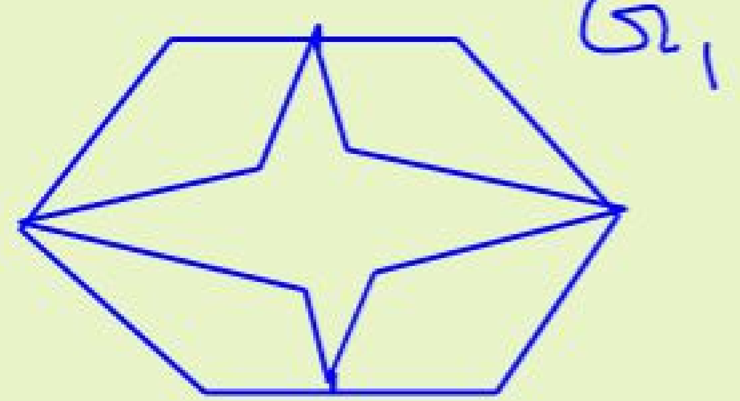

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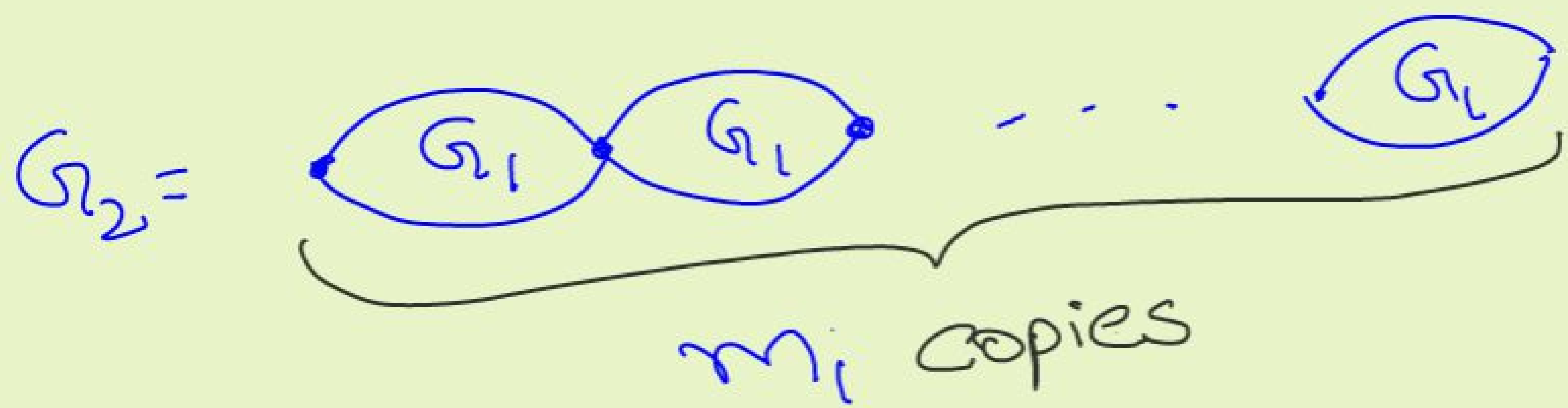

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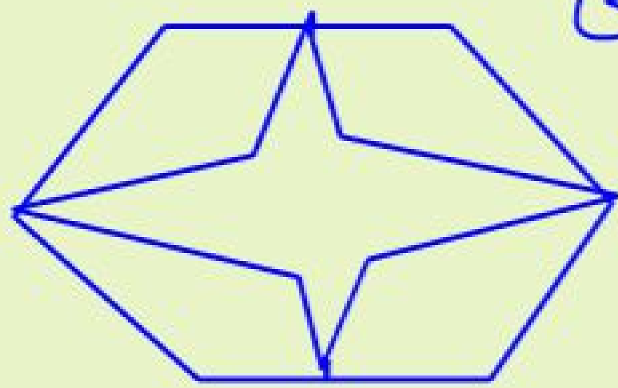

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$G_2 =$

P_2 computed by $\Pi\Sigma\Pi$ formula of size $O(m_1)$.

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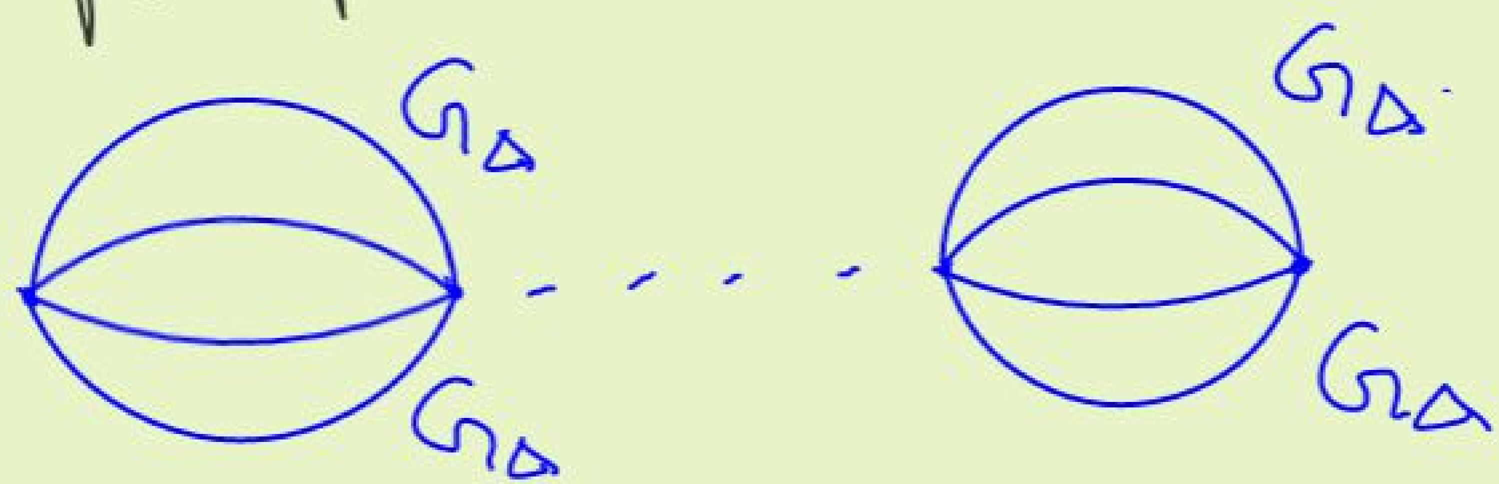

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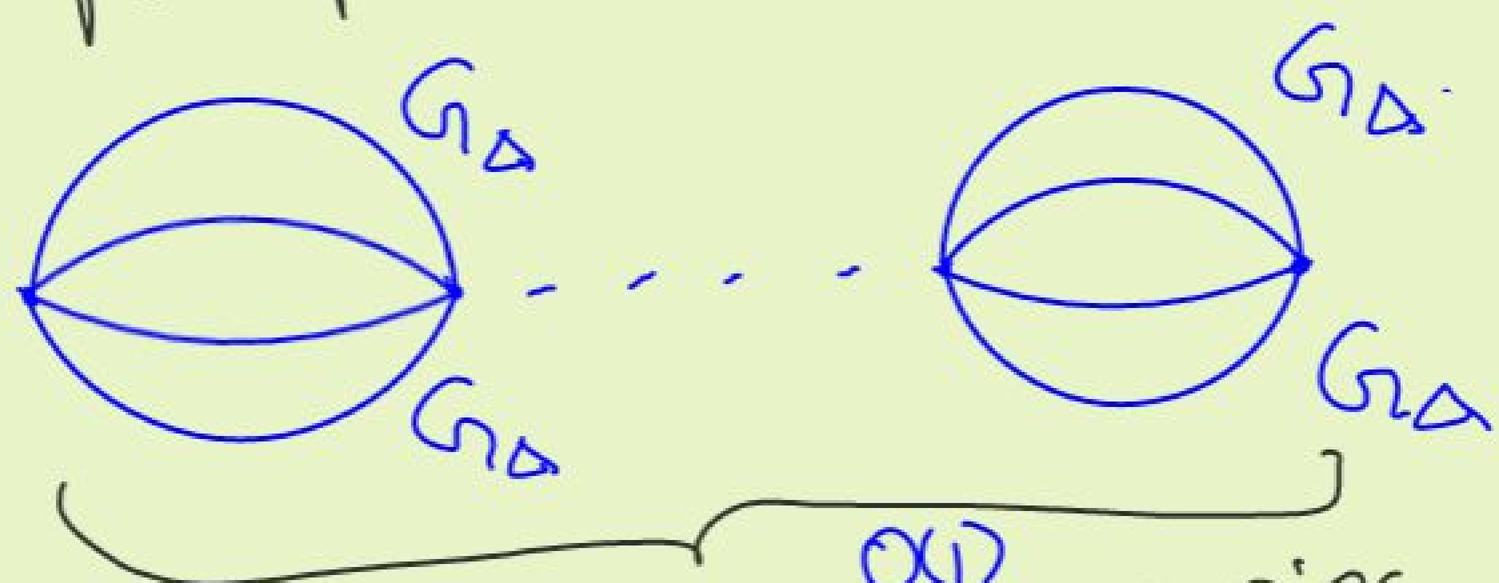

$m_\Delta = n_\Delta^{O(1)}$ copies

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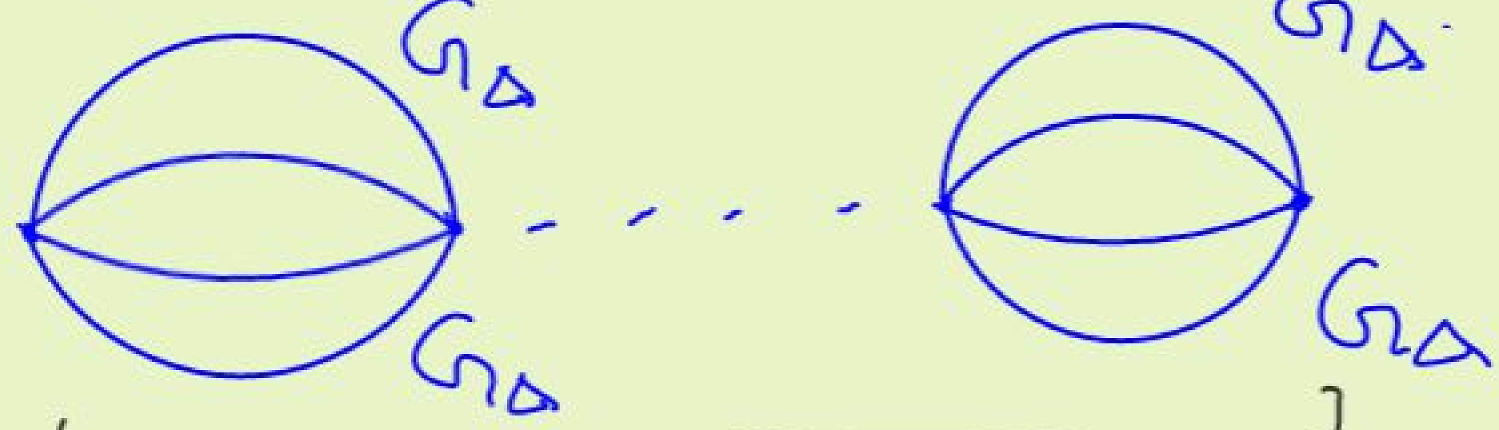

$\underbrace{\qquad\qquad\qquad}_{m_\Delta = n_\Delta^{O(1)}}$ copies

$$P_{\Delta+1} = \prod_{i=1}^{m_\Delta} \left(P_\Delta^{(i,1)} + P_\Delta^{(i,2)} \right)$$

Main thm (restated)

Any p.depth Δ formula for $P_{\Delta+1}$ has size $\exp(\Omega(m_1)) = \exp\left(n_{\Delta+1}^{\Omega(1)}\right)$

Some Proof Details ($\Delta = 1$)

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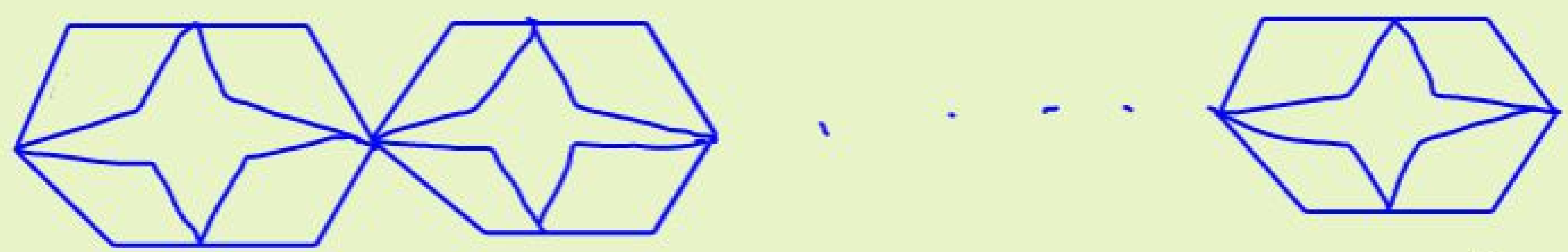
$P_2 =$ 

m_1 copies

<u>Thm</u>: Any $\Sigma\Pi\Sigma$ formula for P_2 must have size $\exp(\Omega(m_1))$.

Raz's rank measure

$P \in \mathbb{F}[Y \cup Z]$

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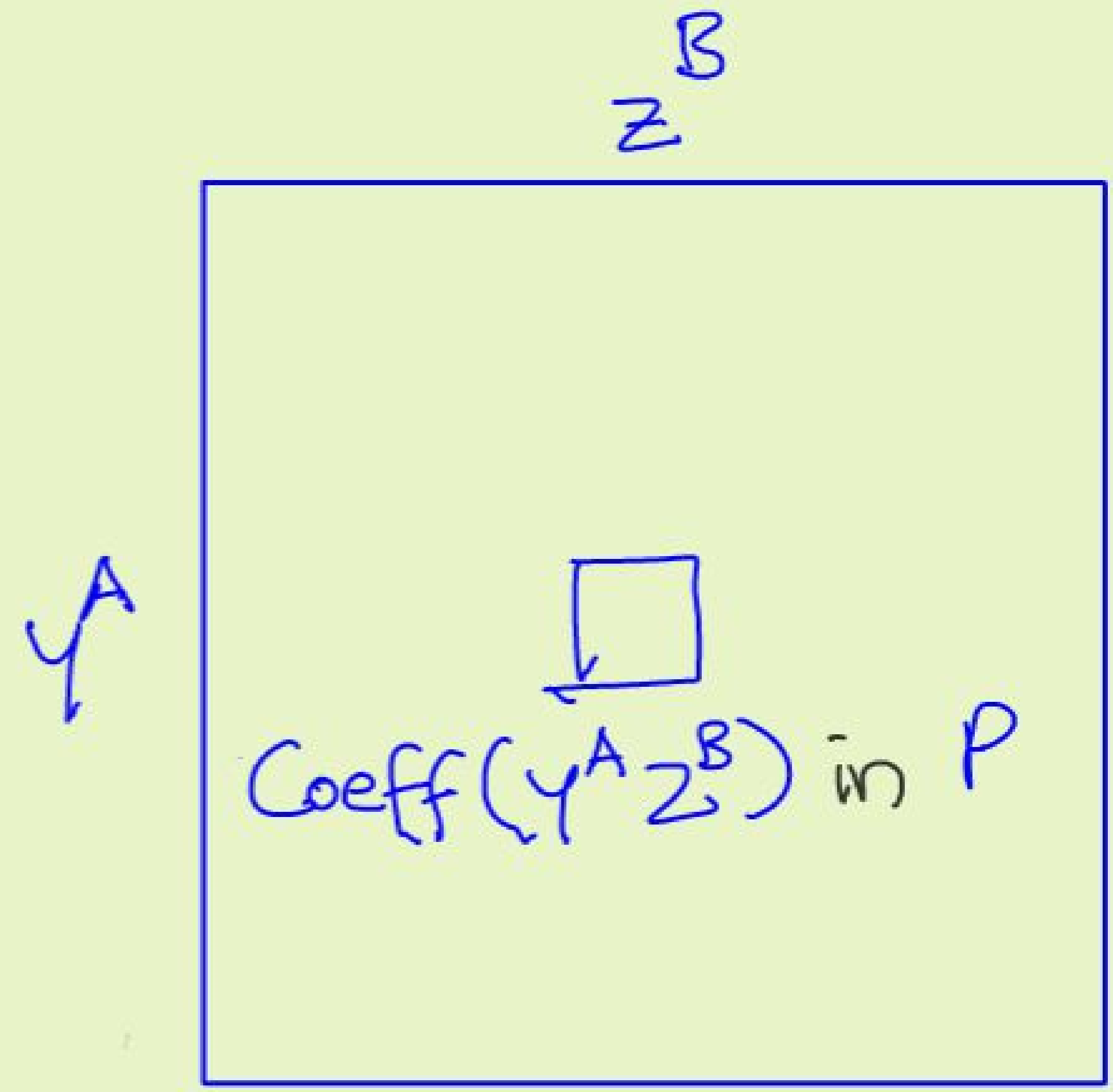

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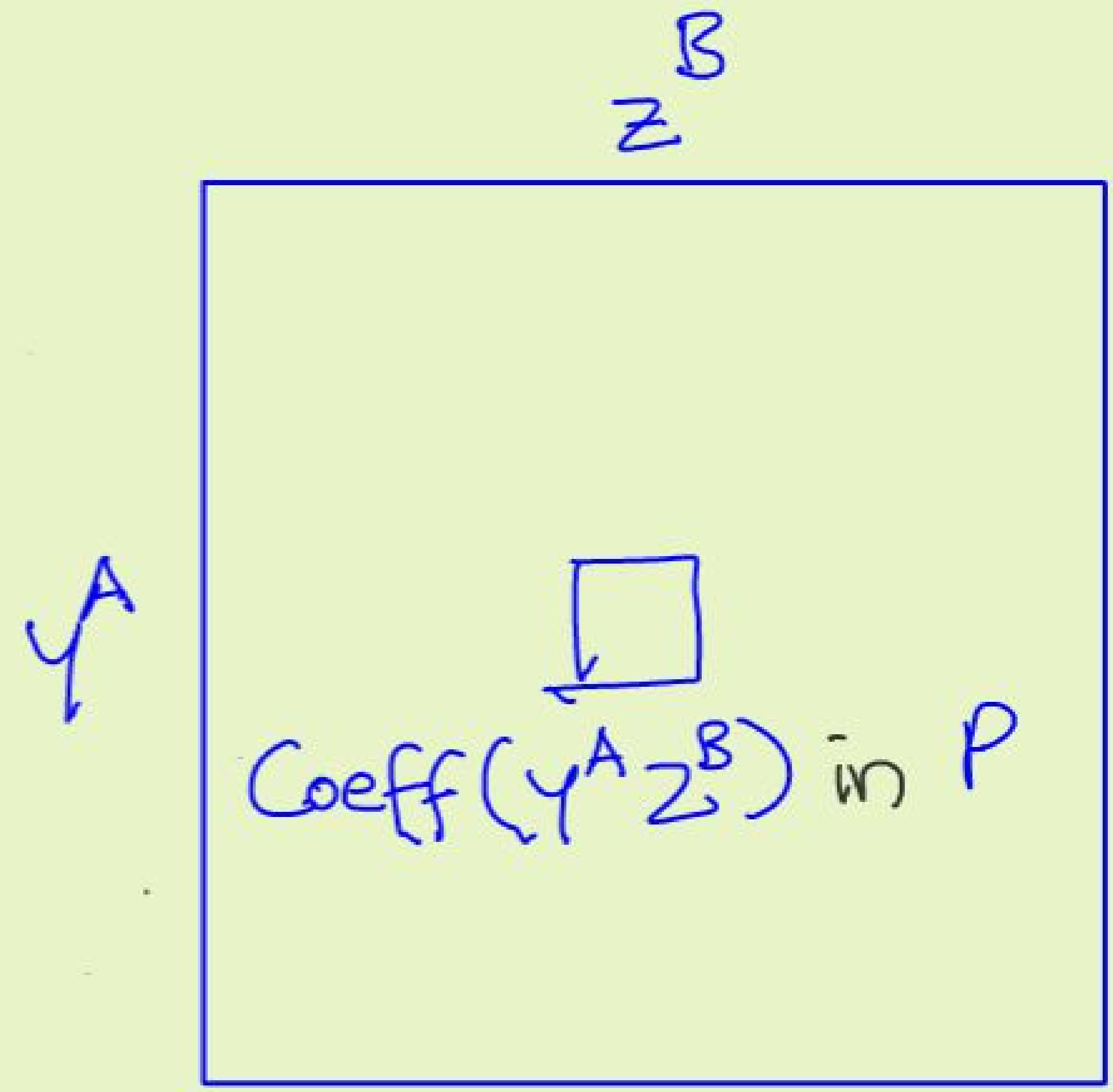
$\leq 2^{\min\{|Y|, |Z|\}}$



Raz's rank measure

$P \in \mathbb{F}[Y \cup Z]$

$M_P \in \mathbb{F}^{2^{|Y|} \times 2^{|Z|}}$

$$M_P = \begin{array}{c|c} & Z^B \\ \hline Y^A & \mathrm{Coeff}(Y^A Z^B) \text{ in } P \end{array}$$

$\mu(P) = \mathrm{rank}(M_P)$

$\leq 2^{\min\{|Y|, |Z|\}}$

<u>Idea</u>: $\mu(\cdot)$ small for small formulas but large for hard polynomial.

A simple example

$$P(y_1, \ldots, y_m, z_1, \ldots, z_m) = \prod_{i=1}^{m} (y_i + z_i)$$

$$= \sum_{A \subseteq [m]} y^A z^{\bar{A}}$$

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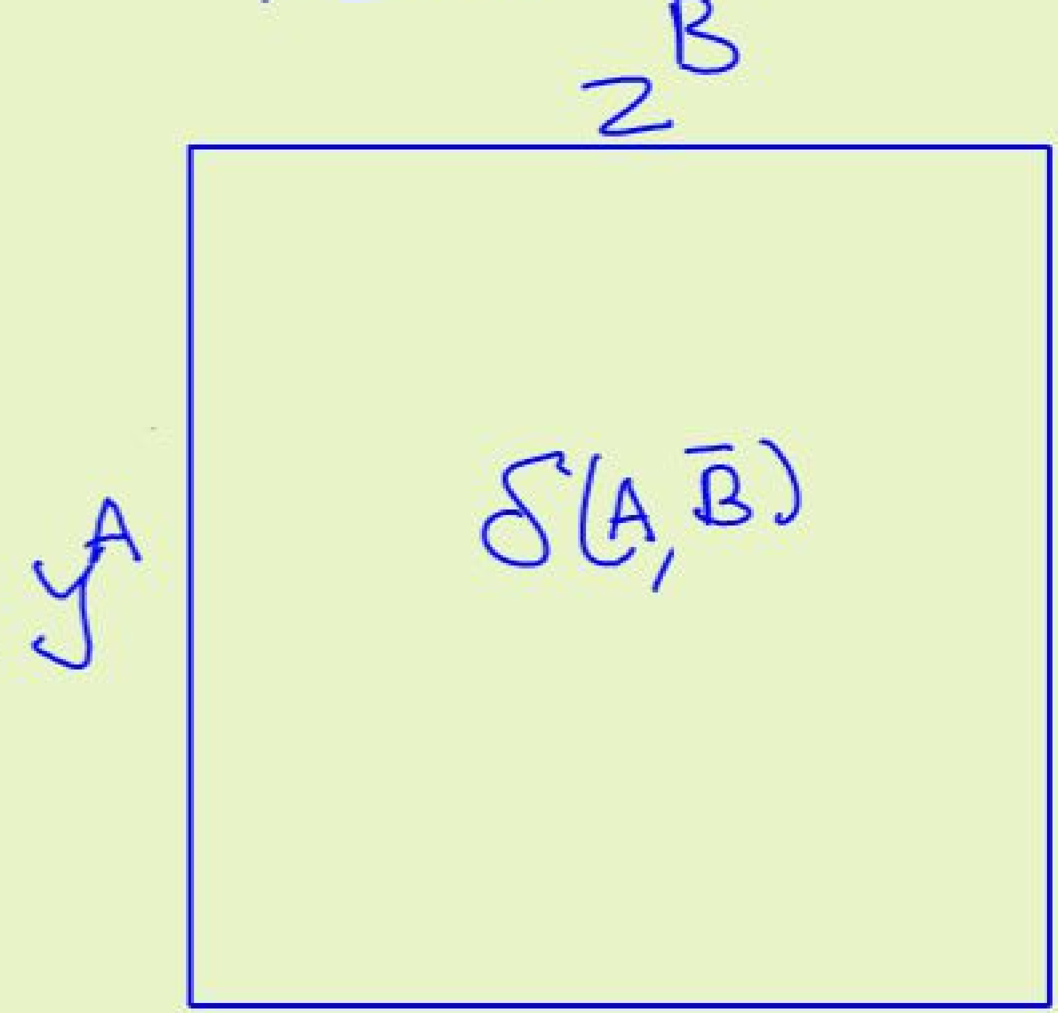
$\mu(\cdot)$ can be large for simple formulas.



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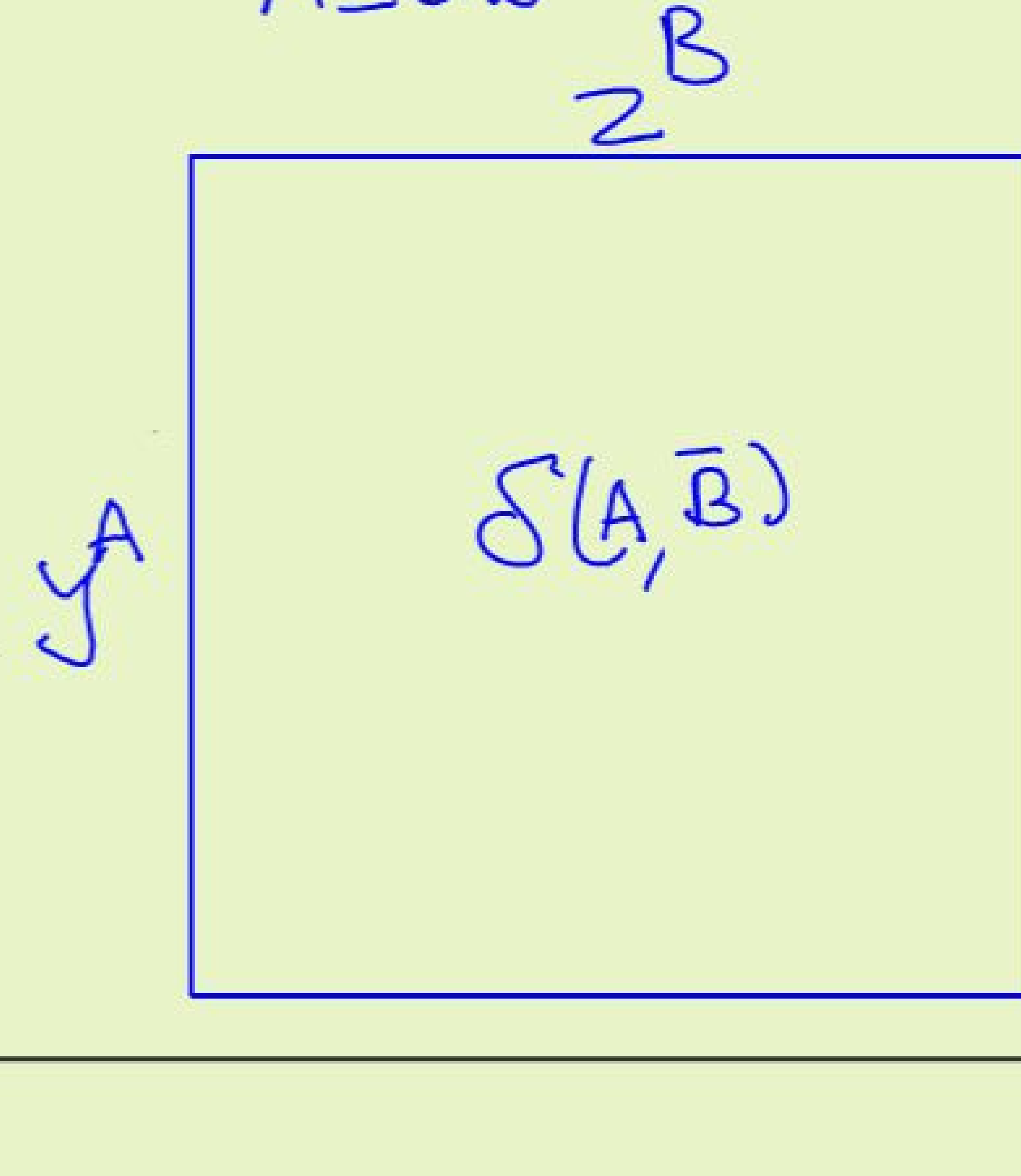

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Fix: but not in a "robust" way.

Restriction

Variable set X

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$\rho : X \to Y \cup Z \cup \mathbb{F}$

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Design a restriction ρ such that

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Design a restriction ρ such that

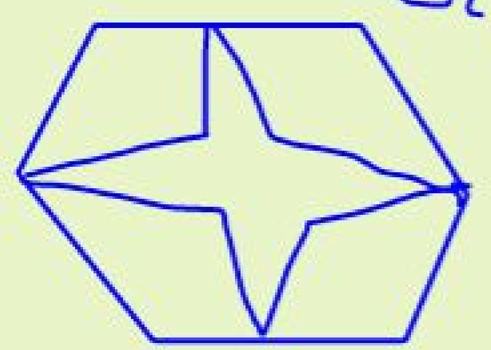
→ $\mu(P_2|_\rho)$ large,

→ $\mu(F|_\rho)$ small for any small $\Sigma\Pi\Sigma$ formula

Random Restriction

$\rho : X \to Y \cup Z \cup \{0,1\}$

G_1



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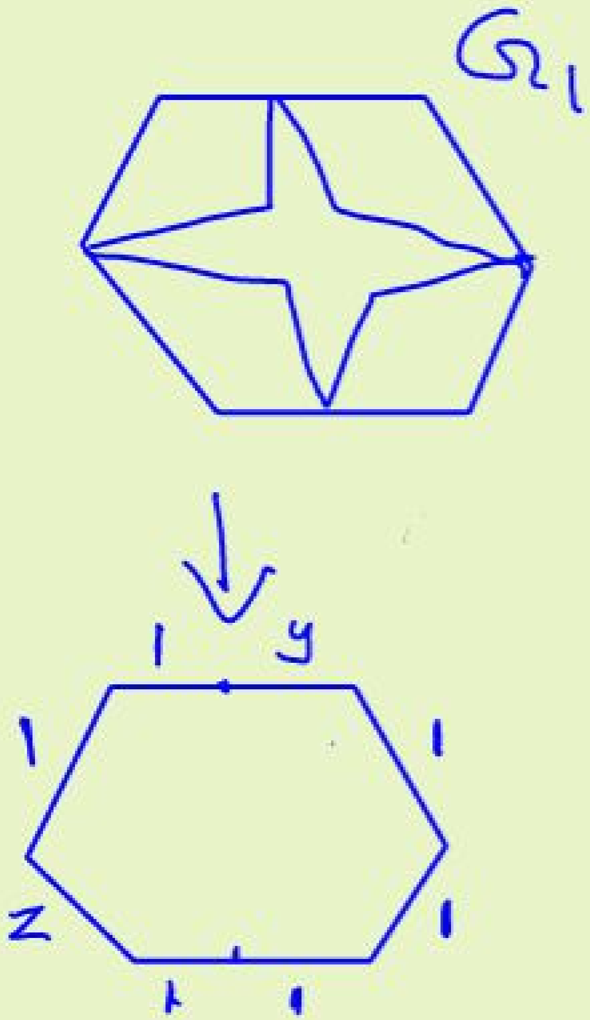

$(y+z)$

Random Restriction

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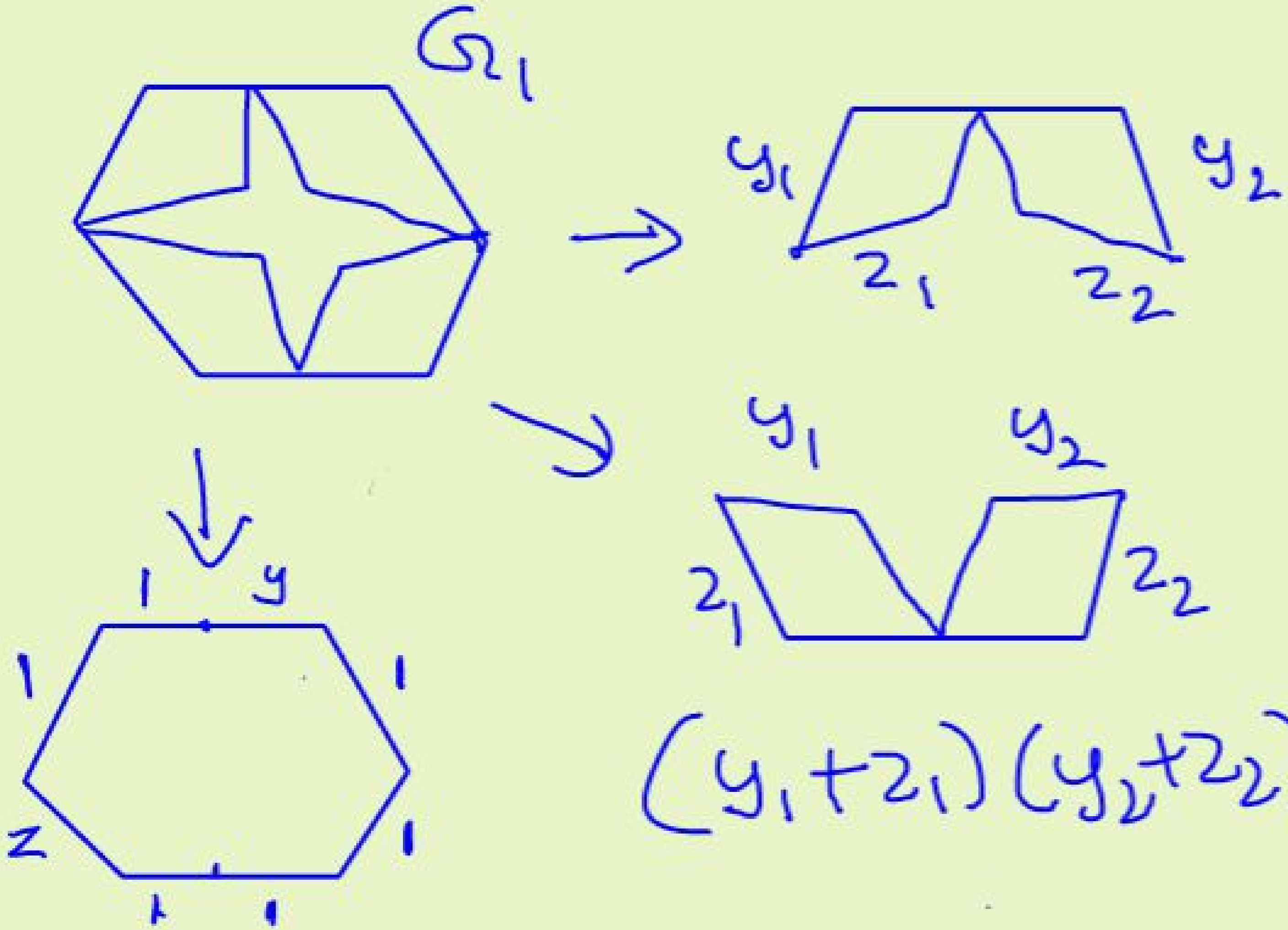

$(y_1+z_1)(y_2+z_2)$

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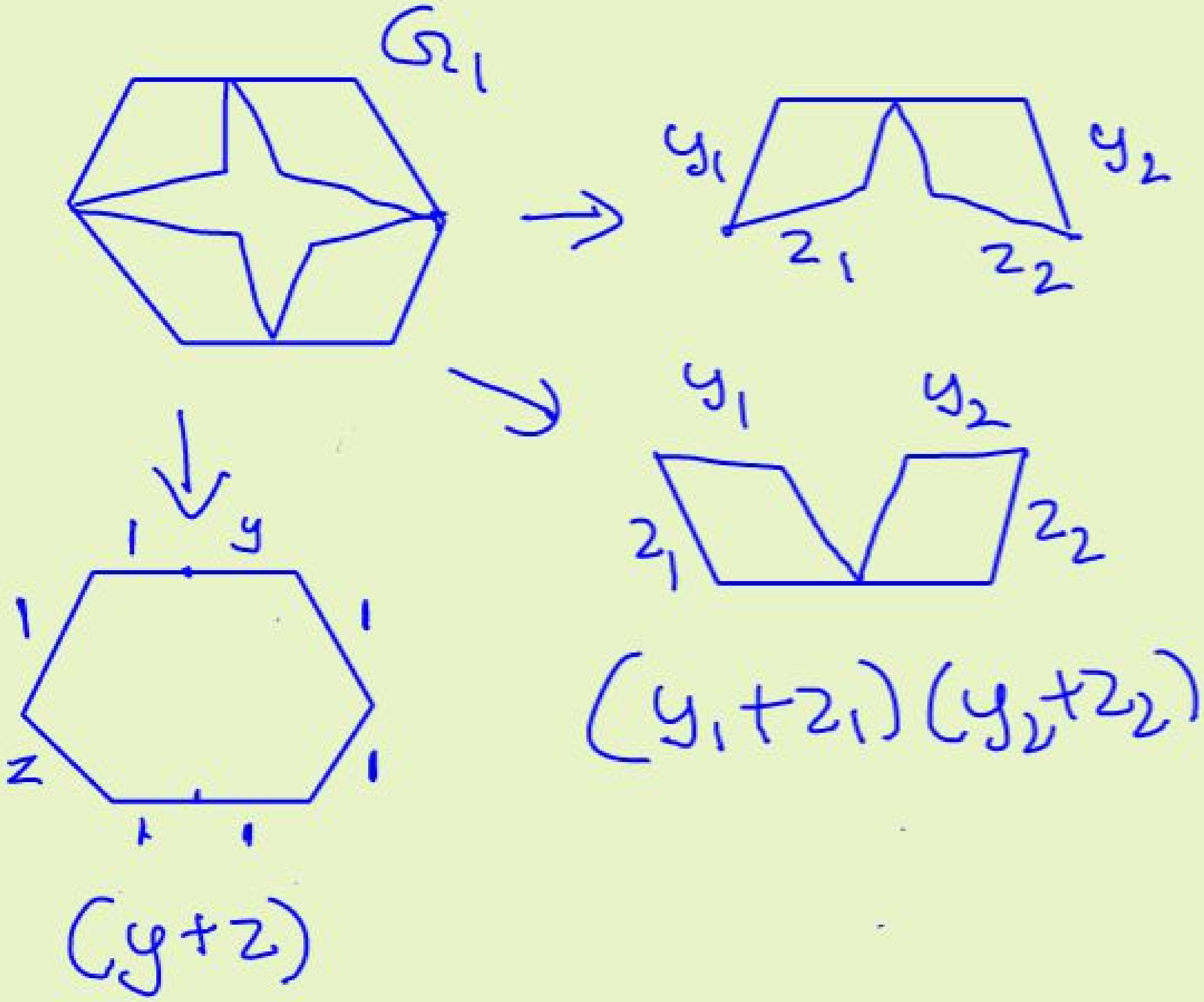

Choose a uniform random option for each G_1.

Random Restriction

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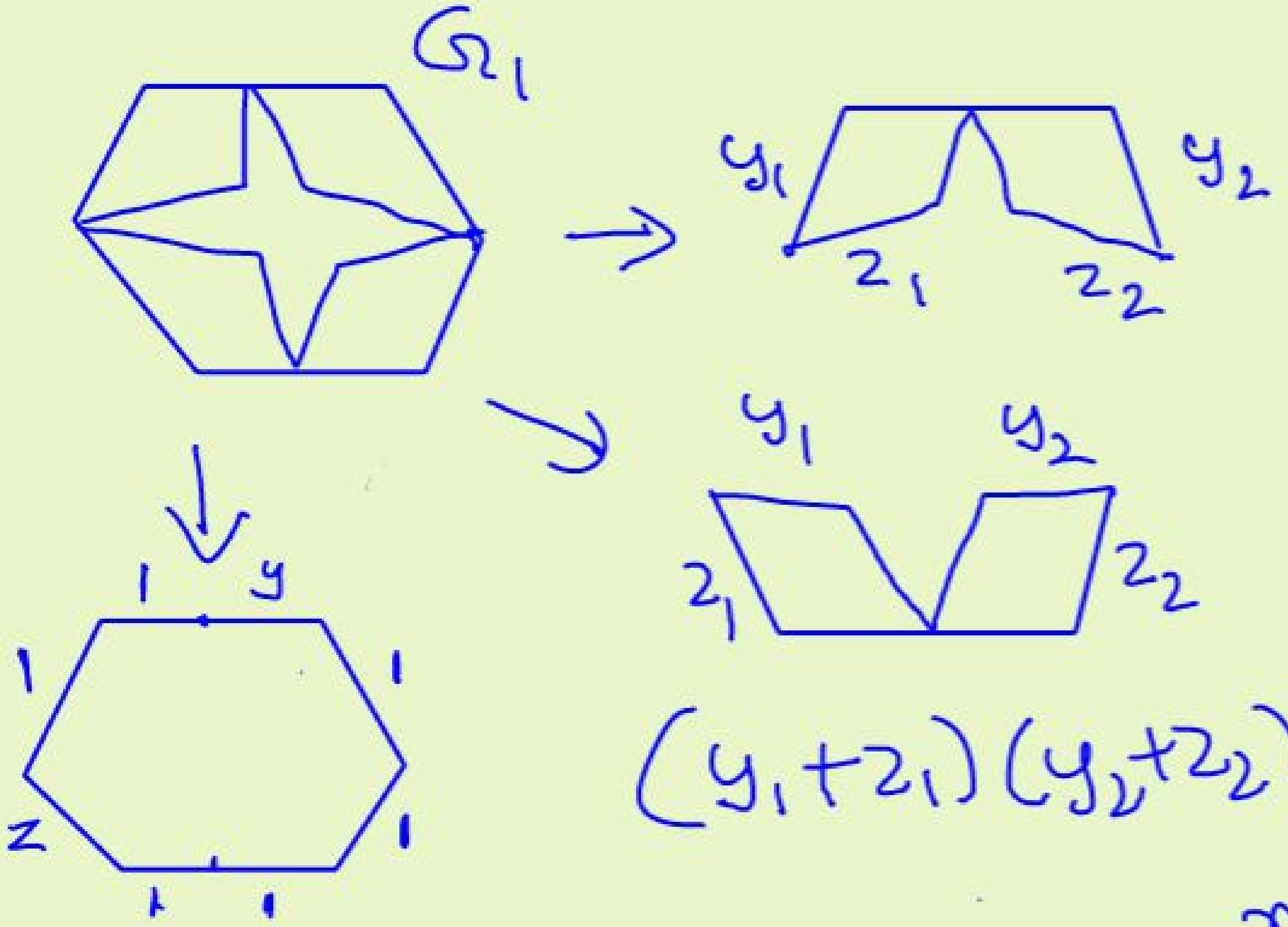

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$$P_2|_\rho = \prod_{i=1}^{m} (y_i + z_i)$$

full rank w.p. 1.

Effect on $\Sigma\Pi\Sigma$ formula

$$F = \sum_{i=1}^{s} \prod_{j=1}^{t} L_{i,j}(x_{i,j}) \quad , \quad X = \bigsqcup_{j} x_{i,j}$$

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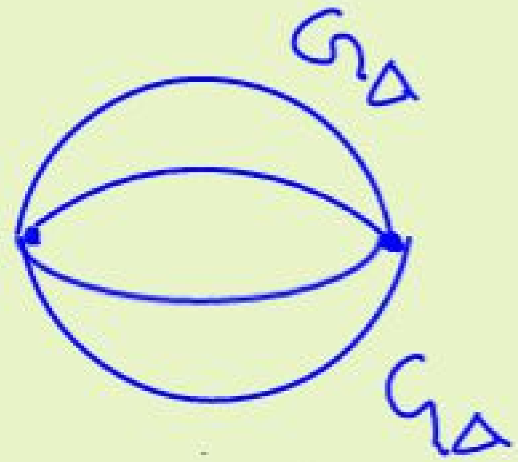
- → Each linear poly. is "rank-deficient" with constant probability.
- → Each product term is highly "rank-deficient" with high probability
- → Subadditivity of rank ⟹ # of terms must be large.

Larger depths

$P_{\Delta+1}: X \to Y \cup Z \cup \{0,1\}$



Larger depths

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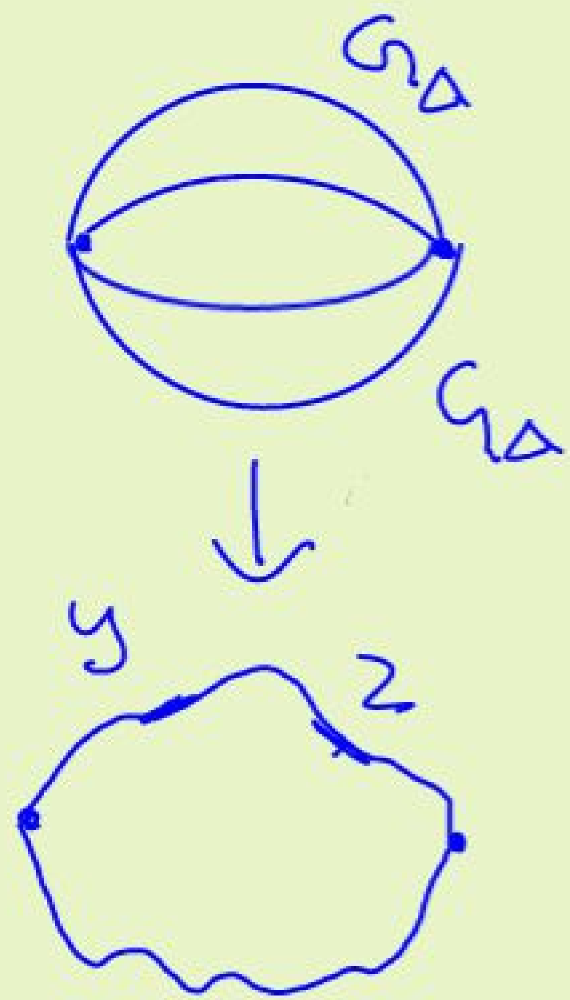

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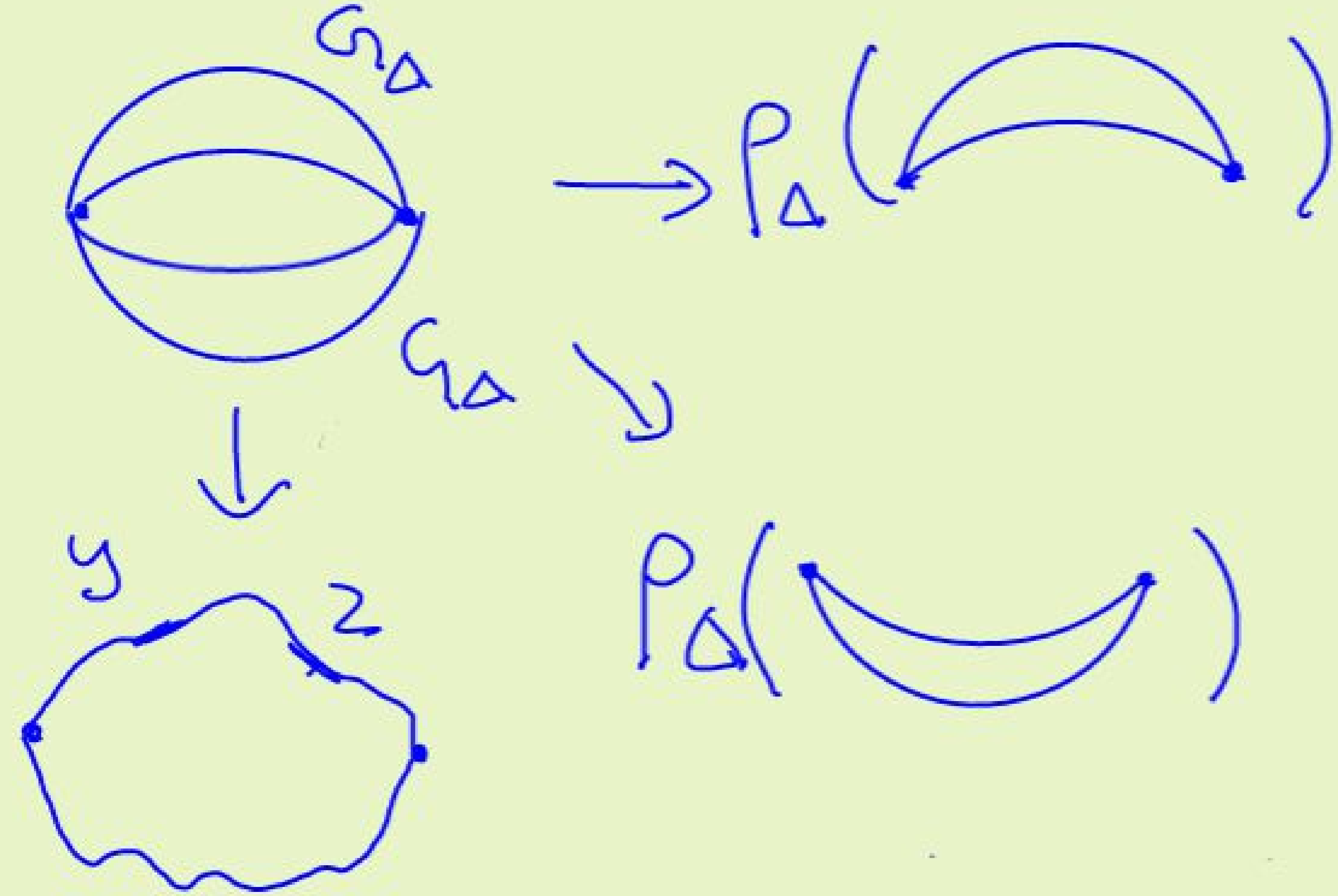

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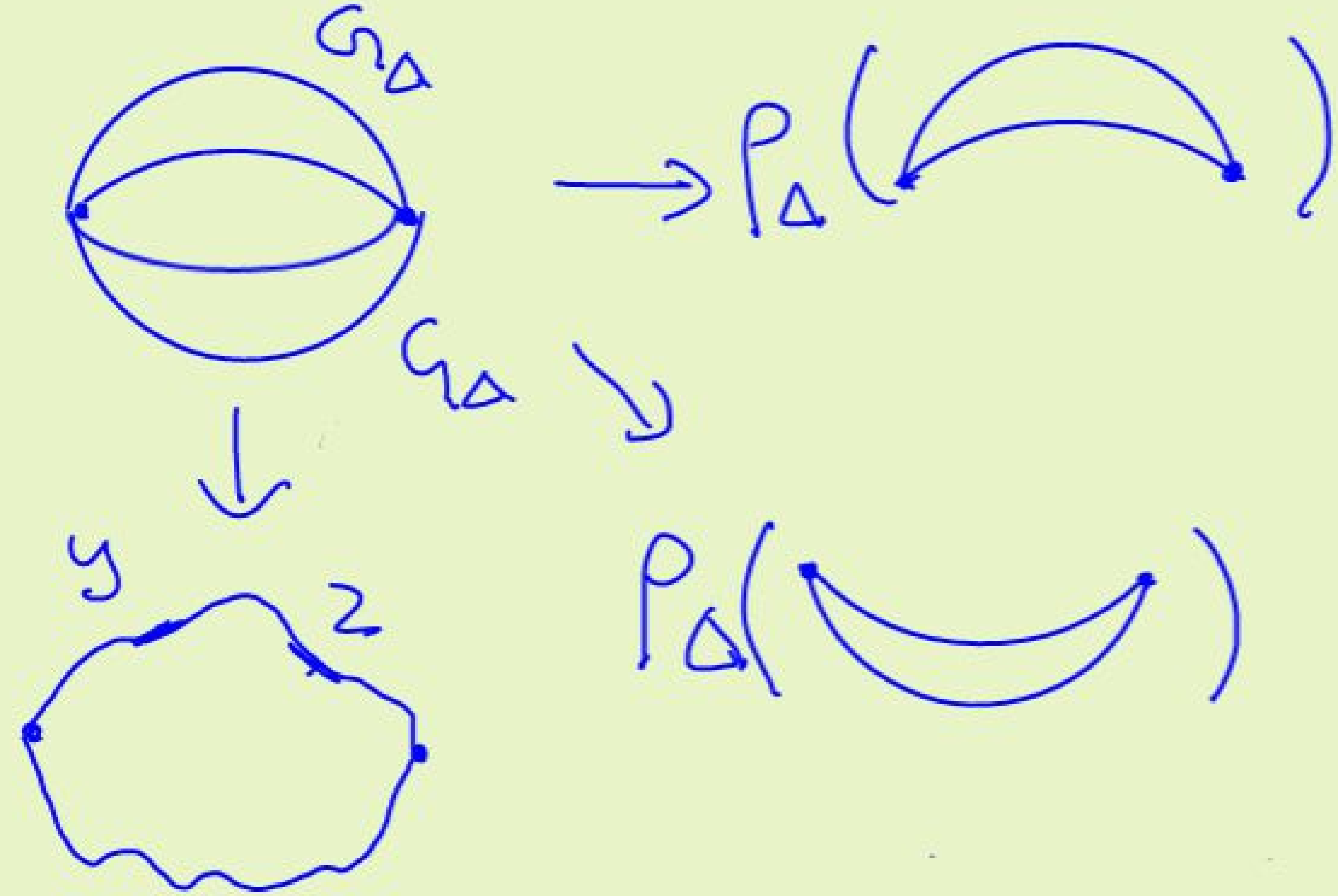

Choose uniformly at random among these options

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$\boxed{\text{Thank you!}}$